On the real entropy of quadratic rational maps

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1. An Overview
   - Entropy in Families of Polynomial Interval Maps
   - Transitioning from Polynomials to Rational Maps

2. The Entropy Function on the Moduli of Real Rational Maps
   - The Moduli Space $\mathcal{M}_d(\mathbb{C})$
   - The Entropy Function $h_\mathbb{R} : \mathcal{M}'_d - S' \to [0, \log(d)]$

3. The Moduli of Real Quadratic Rational Maps
   - The Degree Zero Component of the Domain of $h_\mathbb{R}$
   - Entropy Plots

4. A Monotonicity Result

5. A Non-Monotonicity Result
Entropy of Multimodal Interval or Circle Maps
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Consider a continuous multimodal self-map of a compact interval or a circle.

\[ f : I \quad f : S^1 \]

**Fact (Misiurewicz-Szlenk 1980)**

The topological entropy is the growth rate of the lap number (the number of monotonic pieces) of iterates.

\[
h_{\text{top}}(f) = \lim_{n \to \infty} \frac{1}{n} \log (l(f^{\circ n})) = \inf_n \frac{1}{n} \log (l(f^{\circ n})).
\]

\[ l(f) = 2 \quad l(f^{\circ 2}) = 4 \quad l(f^{\circ 3}) = 6 \]
The Motivating Question

How the topological entropy varies in a family of interval or circle maps?
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- Consider the *Logistic Family*:

\[
\begin{align*}
  f_a : [0, 1] &\to [0, 1] \\
  x &\mapsto ax(1 - x) \quad (a \in [0, 4]).
\end{align*}
\]
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How the topological entropy varies in a family of interval or circle maps?

- Consider the **Logistic Family**: 
  \[ f_a : [0, 1] \to [0, 1] \]
  \[ x \mapsto ax(1 - x) \quad (a \in [0, 4]). \]

- The bifurcation diagram

- The entropy as a function of \( a \)
Two Important Properties

We observe that the function $a \mapsto h_{\text{top}}(f_a)$ is a Devil’s Staircase:
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- It is increasing.

- It is not strictly increasing over any open subinterval.
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**The Monotonicity of Entropy for the Quadratic Family**
[Milnor-Thurston 1988, Douady-Hubbard-Sullivan]

- It is not strictly increasing over any open subinterval.

**The Density of Hyperbolicity for the Quadratic Family** [Lyubich 1997, Graczyk-Świątek 1997]
What About Higher Degrees?

- The meaning of “monotonicity” when the entropy function is multivariate?
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- The meaning of “monotonicity” when the entropy function is multivariate?
  Answer: The connectedness of level sets (the *isentropes*).
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**Monotonicity of Entropy Conjecture (Milnor 1992)**

For any $d \geq 2$ the entropy function is monotonic on the parameter space of degree $d$ “polynomial maps” of the interval.
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- Here we deal with the class of maps $f : [0, 1] \circ s.t.$
  - the map is boundary-anchored $f ([0, 1]) \subseteq \{0, 1\}$ with a fixed orientation;
  - $f$ is a degree $d$ polynomial with $d - 1$ distinct critical points in $(0, 1)$. 

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  ► Critical values then yield an easy description of the parameter space as an open subset of $(0, 1)^{d-1}$. 
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**Theorem (Bruin-van Strien 2009)**

Milnor’s conjecture holds for the aforementioned family of polynomial maps.
The entropy behavior of real quadratic rational maps

The Main Result

The entropy function on the moduli space of real quadratic rational maps is not monotonic, but its restriction to certain dynamically defined regions is monotonic.
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The entropy function on the moduli space of real quadratic rational maps is not monotonic, but its restriction to certain dynamically defined regions is monotonic.

Things that should be addressed before the proof:
- the “real entropy” of rational maps;
- “real entropy function” defined on a “moduli space” of real rational maps;
- the structure of the moduli space of quadratic rational maps;
- the experimental evidence on which this result is based.
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The Topological Entropy of a Rational Map

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Fact (Lyubich 1981)

The topological entropy of rational map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of degree $d \geq 2$ is $\log(d)$. 
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So we need to focus on the entropy of a subsystem: We consider rational maps with real coefficients:

$$f \in \mathbb{C}(z) \text{ with real coefficients} \iff f \in \mathbb{C}(z) \text{ keeps } \hat{\mathbb{R}} := \mathbb{R} \cup \{\infty\} \text{ invariant.}$$
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**Definition**

The *Real Entropy* of \( f \in \mathbb{R}(z) \) is defined as:

\[
h_{\mathbb{R}}(f) := h_{\text{top}} \left( f \big|_{\hat{\mathbb{R}}} : \hat{\mathbb{R}} \to \hat{\mathbb{R}} \right) \in [0, \log(\deg f)].
\]
Subtleties Compared to the Polynomial Case
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What can be said about the level sets of the function $h_{\mathbb{R}}$ on the “space” of real rational maps of degree $d$?

1. Rather than boundary-anchored interval maps one has to deal with circle maps $f \upharpoonright_{\hat{\mathbb{R}}}: \hat{\mathbb{R}} \circlearrowright$. 
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Question

What can be said about the level sets of the function $h_R$ on the “space” of real rational maps of degree $d$?

1. Rather than boundary-anchored interval maps one has to deal with circle maps $f \upharpoonright \mathbb{R}: \mathbb{R} \to \mathbb{R}$.

2. This context is far more general: The maps $f \upharpoonright \mathbb{R}: \mathbb{R} \to \mathbb{R}$ come with various lap numbers and topological degrees.
Subtleties Compared to the Polynomial Case

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2. This context is far more general: The maps $f \upharpoonright_{\mathbb{R}} : \hat{\mathbb{R}} \circlearrowleft$ come with various lap numbers and topological degrees.

3. How should we parametrize real rational maps of degree $d$?
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The Moduli Space of Degree $d$ Rational Maps

- The *Moduli Space* of rational maps of degree $d \geq 2$ is a complex variety of dimension $2d - 2$:

\[
\mathcal{M}_d(\mathbb{C}) := \frac{\text{Rat}_d(\mathbb{C})}{\text{PSL}_2(\mathbb{C})} = \left\{ f \in \mathbb{C}(z) \text{ rational of degree } d \right\} \quad (\alpha \in \text{PSL}_2(\mathbb{C}))
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\]

- We consider the complex conjugacy classes of real maps:

\[
\mathcal{M}'_d := \frac{\text{PSL}_2(\mathbb{C}). \left( \text{Rat}_d(\mathbb{R}) \right)}{\text{PSL}_2(\mathbb{C})} \quad \mathcal{M}'_d \subset \mathcal{M}_d(\mathbb{C})
\]
Main Example: The Moduli Space $\mathcal{M}_2$

The moduli space of quadratic rational maps can be identified with the plane $\mathbb{C}^2$ [Milnor 1993] via

$$\langle f \rangle \mapsto (\sigma_1 = \lambda_1 + \lambda_2 + \lambda_3, \sigma_2 = \lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1)$$
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where

$\lambda_1, \lambda_2, \lambda_3 :$ the multipliers of fixed points (Möbius invariants)

satisfy the fixed point formula:

$$\frac{1}{1 - \lambda_1} + \frac{1}{1 - \lambda_2} + \frac{1}{1 - \lambda_3} = 1.$$
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- $\mathcal{M}_2' \cong \mathbb{R}^2$ is the underlying real $(\sigma_1, \sigma_2)$-plane.
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$h_{\mathbb{R}}$ as a Function on $\mathcal{M}'$?!
On the real entropy of quadratic rational maps

The Entropy Function on the Moduli of Real Rational Maps

The Entropy Function $h_R : \mathcal{M}_d' \to S' \to [0, \log(d)]$

$h_R$ as a Function on $\mathcal{M}_d'$.?!\

First Attempt

$$\langle f \rangle \mapsto h_R(f) = h_{\text{top}} \left( f \upharpoonright \hat{\mathbb{R}} : \hat{\mathbb{R}} \to \hat{\mathbb{R}} \right)$$
$h_{\mathbb{R}}$ as a Function on $\mathcal{M}'_d$?!

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$$\langle f \rangle \mapsto h_{\mathbb{R}}(f) = h_{\text{top}} \left( f \upharpoonright \hat{\mathbb{R}} : \hat{\mathbb{R}} \to \hat{\mathbb{R}} \right)$$

not well defined!

There might be representatives conjugate only by elements of $\text{PSL}_2(\mathbb{C}) = \text{PGL}_2(\mathbb{C})$, not by elements of $\text{PGL}_2(\mathbb{R})$. 
The Entropy Function $h_{\mathbb{R}} : \mathcal{M}'_d - S' \to [0, \log(d)]$

**$h_{\mathbb{R}}$ as a Function on $\mathcal{M}'_d$?!**

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**Example**

Maps $z \mapsto \frac{1}{\mu} \left( z \pm \frac{1}{z} \right) \ (\mu \in \mathbb{R} - \{0\})$ are conjugate over $\mathbb{C}$

$$\frac{1}{i} \cdot \frac{1}{\mu} \left( iz + \frac{1}{iz} \right) = \frac{1}{\mu} \left( z - \frac{1}{z} \right);$$

but restricted to $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$:

- $x \mapsto \frac{1}{\mu} \left( x - \frac{1}{x} \right)$ is a two-sheeted covering and of entropy $\log(2)$;
- $x \mapsto \frac{1}{\mu} \left( x + \frac{1}{x} \right)$ is of entropy zero.
Excluding Symmetries

This issue of real rational maps that are conjugate only over complex numbers can happen only in presence of Möbius symmetries; e.g.

\[-\frac{1}{\mu} \left( z + \frac{1}{z} \right) = \frac{1}{\mu} \left( -z + \frac{1}{-z} \right).\]
Excluding Symmetries

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\[-\frac{1}{\mu} \left( z + \frac{1}{z} \right) = \frac{1}{\mu} \left( (-z) + \frac{1}{-z} \right).\]

So we have to exclude the symmetry locus

\[S' := \left\{ \langle f \rangle \in \mathcal{M}_d(\mathbb{C}) \mid f \in \text{Rat}_d(\mathbb{R}), \text{Aut}(f) \neq \{1\} \right\} \subset \mathcal{M}'_d\]

in order to have a well defined real entropy function.
The Real Entropy Function $h^{}_\mathbb{R}$

Proposition (F. 2018)

For any $d \geq 2$:
- $\mathcal{M}'_d - S'$ is an irreducible real variety of dimension $2d - 2$.
- The function

$$
\begin{cases}
  h^{}_{\mathbb{R}} : \mathcal{M}'_d - S' \to [0, \log(d)] \\
  \langle f \rangle \mapsto h^{}_{\text{top}} \left( f \mid^{}_{\mathring{\mathbb{R}} : \mathring{\mathbb{R}} \to \mathring{\mathbb{R}}} \right) \\
\end{cases}
\quad (f \in \mathbb{R}(z))
$$

is surjective and continuous (in the analytic topology).
- The domain is disconnected with components (each of the same real dimension $2d - 2$) corresponding to possible topological degrees of the restriction $f \mid^{}_{\mathring{\mathbb{R}} : \mathring{\mathbb{R}} \to \mathring{\mathbb{R}}}$.
The Real Entropy Function $h_R$.

**Proposition (F. 2018)**

For any $d \geq 2$:

- $\mathcal{M}_d' - S'$ is an irreducible real variety of dimension $2d - 2$.
- The function
  \[
  \begin{cases}
  h_R : \mathcal{M}_d' - S' \to [0, \log(d)] \\
  \langle f \rangle \mapsto h_{\text{top}} \left( f \restriction_{\hat{\mathbb{R}}} : \hat{\mathbb{R}} \to \hat{\mathbb{R}} \right) 
  \end{cases}
  \]
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  is surjective and continuous (in the analytic topology).

- The domain is disconnected with components (each of the same real dimension $2d - 2$) corresponding to possible topological degrees of the restriction $f \restriction_{\hat{\mathbb{R}}} : \hat{\mathbb{R}} \to \hat{\mathbb{R}}$.

**The Monotonicity Question**

Restricted to connected components of the domain, are the level sets of $h_R$ connected?
Back to the Main Example: $\mathcal{M}_2' - S'$ has three connected components.

$S' = \left\{ \left\langle \frac{1}{\mu} \left( z + \frac{1}{z} \right) \right\rangle = \left\langle \frac{1}{\mu} \left( z - \frac{1}{z} \right) \right\rangle \right\vert \mu \in \mathbb{R} - \{0\} \right\}$ a cubic curve in $\mathcal{M}_2' = \mathbb{R}^2$

The topological degree of the restriction is $\pm 2$ or zero.
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A Simple Dichotomy for Real Quadratic Maps
A Simple Dichotomy for Real Quadratic Maps

For $f \in \mathbb{R}(z)$ of degree two; either

- the two critical points are complex conjugate $\Rightarrow f \rvert_{\hat{\mathbb{R}}}: \hat{\mathbb{R}} \to \hat{\mathbb{R}}$ a 2-sheeted covering $\Rightarrow h_{\mathbb{R}}(f) = \log(2)$

- or the critical points are real; $f(\hat{\mathbb{R}}) \subsetneq \hat{\mathbb{R}} \Rightarrow$ only the interval map $f \rvert_{f(\hat{\mathbb{R}})}: f(\hat{\mathbb{R}}) \to f(\hat{\mathbb{R}})$ matters dynamically.
A Simple Dichotomy for Real Quadratic Maps

For $f \in \mathbb{R}(z)$ of degree two; either

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Upshot $\star$ Among all three connected components of $\mathcal{M}^{'}_{2} - S^{'}$ only one connected component is relevant to our discussion; the component of degree zero maps.
The Component of Degree Zero Maps in $\mathcal{M}'_2 - S'$

- There is a finer partition of this component according to the orientation and modality of the interval map.
- $h_R \equiv 0$ on monotonic regions and $h_R \equiv \log(2)$ on deg $\pm 2$ regions.
- Upshot: Only the unimodal region and the two bimodal regions adjacent to it matter to the monotonicity discussion.
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- $h_R \equiv 0$ on monotonic regions and $h_R \equiv \log(2)$ on $\deg \pm 2$ regions.

- **Upshot** *Only the unimodal region and the two bimodal regions adjacent to it matter to the monotonicity discussion.*
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An Entropy Contour Plot in the Unimodal and \((- + -)\)-Bimodal Regions

\[\begin{align*}
\text{black} & \quad [0, 0.1) \\
\text{blue} & \quad [0.1, 0.25) \\
\text{magenta} & \quad [0.25, 0.4) \\
\text{green} & \quad [0.4, 0.48) \\
\text{cyan} & \quad [0.48, 0.55) \\
\text{yellow} & \quad [0.55, 0.65) \\
\text{red} & \quad [0.65, \log(2) \approx 0.7]
\end{align*}\]
An Entropy Contour Plot in the Unimodal and \((- + -)\)-Bimodal Regions

Conjecture

*Restricted to the union of adjacent unimodal and \((- + -)\)-bimodal regions, the entropy function is monotonic.*
An Entropy Contour Plot in the (+ − +)-Bimodal Region

black blue magenta green cyan yellow red
failure [0, 0.05) [0.05, 0.2) [0.2, 0.3) [0.3, 0.5) [0.5, 0.66) [0.66, log(2) ≈ 0.7]
An Entropy Contour Plot in the (+ − +)-Bimodal Region

Conjecture

The monotonicity fails here due to a “non-polynomial” behavior.
The Algorithms Used to Generate the Plots

Going back to the moduli space

- Summarizing the conjectures:
Going back to the moduli space

- Summarizing the conjectures:

- An important line in the picture: $\sigma_2 = 2\sigma_1 - 3$ – the locus where one of the fixed points becomes multiple.
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The Statement of the Theorem

Theorem (F. 2018)

Restricted to the part of the moduli space where the critical points are real and the maps admit three real fixed points, the level sets of $h_\mathbb{R}$ are connected.
Proof; Step 1: An Analysis of Real Fixed Points

If there are three real fixed points, at least one of them must be attracting:

\[ \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R} - \{1\} \]
with at least one of them non-negative

\[ 1 - \lambda_1 + 1 - \lambda_2 + 1 - \lambda_3 = 1. \]
Proof; Step 1: An Analysis of Real Fixed Points

If there are three real fixed points, at least one of them must be attracting:

$$\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R} - \{1\} \text{ w/ at least one of them non-negative}$$

$$\Rightarrow \exists i \text{ s.t. } |\lambda_i| < 1.$$
Proof; Step 2: A Straightening Argument
Proof; Step 2: A Straightening Argument

- Douady and Hubbard’s Theory of Polynomial-like Mappings: The fixed point can be made super-attracting by a quasi-conformal perturbation outside its basin.
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- The monotonicity of entropy for quadratic polynomials has been established by Milnor and Thurston.
1. An Overview
   - Entropy in Families of Polynomial Interval Maps
   - Transitioning from Polynomials to Rational Maps

2. The Entropy Function on the Moduli of Real Rational Maps
   - The Moduli Space $\mathcal{M}_d(\mathbb{C})$
   - The Entropy Function $h_\mathbb{R} : \mathcal{M}'_d - S' \rightarrow [0, \log(d)]$

3. The Moduli of Real Quadratic Rational Maps
   - The Degree Zero Component of the Domain of $h_\mathbb{R}$
   - Entropy Plots

4. A Monotonicity Result

5. A Non-Monotonicity Result
An Interesting Bifurcation Behavior

The bifurcation diagram for a part of the \((+-+)-\)bimodal region parametrized as \(\left\{ x \mapsto \frac{2\mu x(t x + 2)}{\mu^2 x^2 + (t x + 2)^2} : [-1, 1] \right\} \). A period-doubling bifurcation from a 5-cycle to a 10-cycle visible as the transition from green to magenta occurs in “various” directions.
A Non-Polynomial Behavior

Theorem (F.-Pilgrim 2019)

The restriction of $h_R$ to the $(+ - +)$-bimodal region admits a continuum of disconnected level sets.
A Non-Polynomial Behavior

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  Bicritical rational maps whose fixed points are all repelling are called *essentially non-polynomial-like* [Milnor-2000].
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- Why “non-polynomial”? Bicritical rational maps whose fixed points are all repelling are called _essentially non-polynomial-like_ [Milnor-2000].
- The main idea of the proof: Construct a family of real hyperbolic components consisting of essentially non-polynomial quadratic rational maps in the \((+ - +)\)-region.