On the real entropy of quadratic rational maps

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Entropy of Multimodal Interval or Circle Maps

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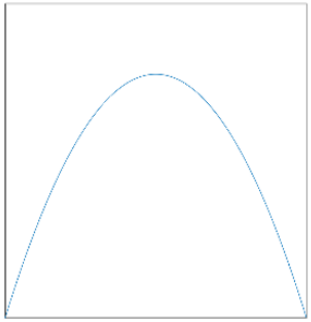
Consider a continuous multimodal self-map of a compact interval or a circle.

$$f : I \circlearrowleft \qquad f : S^1 \circlearrowleft$$

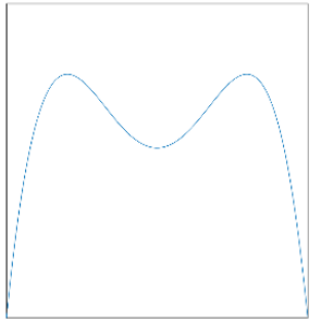
Fact (Misiurewicz-Szlenk 1980)

The topological entropy is the growth rate of the lap number (the number of monotonic pieces) of iterates.

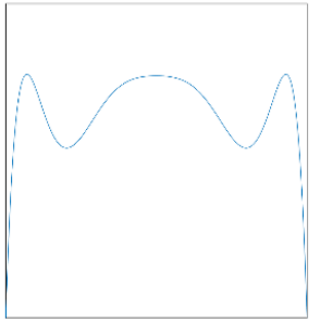
$$h_{\text{top}}(f) = \lim_{n\to\infty} \frac{1}{n} \log\left(l(f^{\circ n})\right) = \inf_n \frac{1}{n} \log\left(l(f^{\circ n})\right).$$



$l(f) = 2$



$l(f^{\circ 2}) = 4$



$l(f^{\circ 3}) = 6$

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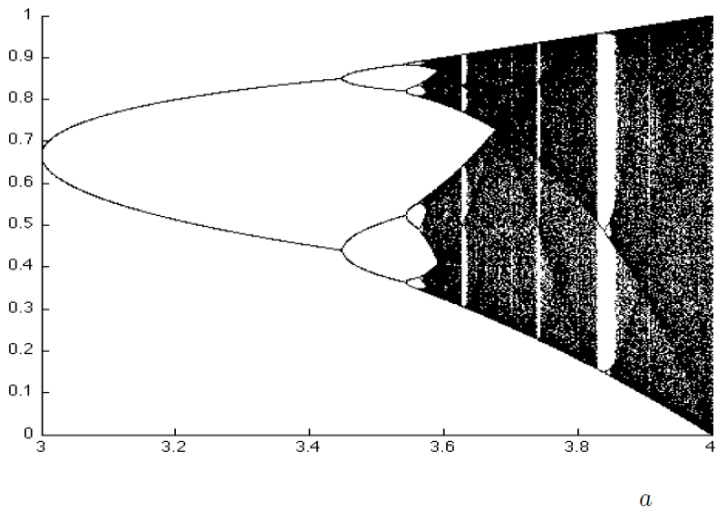
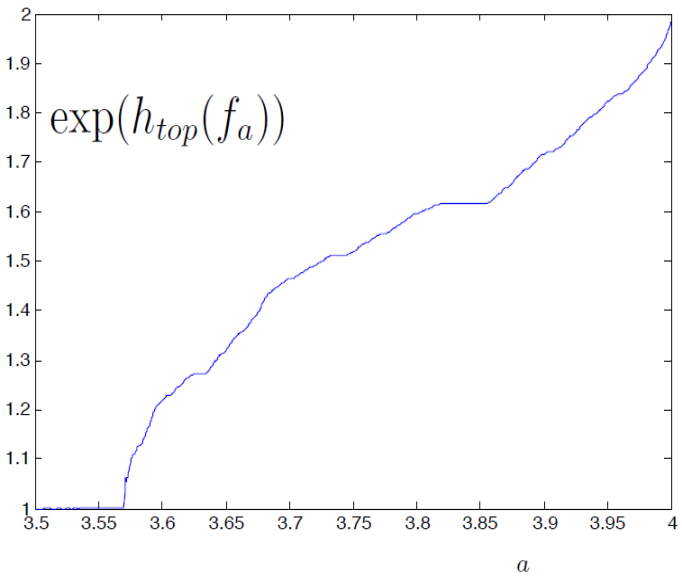
- The bifurcation diagram



The entropy as a function of a



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  The Monotonicity of Entropy for the Quadratic Family [Milnor-Thurston 1988, Douady-Hubbard-Sullivan]

- It is not strictly increasing over any open subinterval.

  The Density of Hyperbolicity for the Quadratic Family [Lyubich 1997, Graczyk-Świątek 1997]

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Theorem (Bruin-van Strien 2009)

Milnor's conjecture holds for the aforementioned family of polynomial maps.

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  - “real entropy function” defined on a “moduli space” of real rational maps;
  - the structure of the moduli space of quadratic rational maps;
  - the experimental evidence on which this result is based.

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So we need to focus on the entropy of a subsystem: We consider rational maps with real coefficients:

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Definition

The *Real Entropy* of $f \in \mathbb{R}(z)$ is defined as:

$$h_{\mathbb{R}}(f) := h_{\text{top}}\left(f \restriction_{\hat{\mathbb{R}}} : \hat{\mathbb{R}} \to \hat{\mathbb{R}}\right) \in [0, \log(\deg f)] \,.$$

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2. This context is far more general: The maps $f\restriction_{\hat{\mathbb{R}}}: \hat{\mathbb{R}} \circlearrowleft$ come with various lap numbers and topological degrees.
3. How should we parametrize real rational maps of degree d?

The Moduli Space of Degree d Rational Maps

- The *Moduli Space* of rational maps of degree $d \geq 2$ is a complex variety of dimension $2d - 2$:

The "space" of Möbius conjugacy classes

$$\mathcal{M}_d(\mathbb{C}) := \mathrm{Rat}_d(\mathbb{C})/\mathrm{PSL}_2(\mathbb{C})$$
$$= \frac{\{f \in \mathbb{C}(z) \text{ rational of degree } d\}}{f \sim \alpha \circ f \circ \alpha^{-1}} \quad (\alpha \in \mathrm{PSL}_2(\mathbb{C}))$$

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- We consider the complex conjugacy classes of real maps:

The subspace of Möbius conjugacy classes of real maps

$$\mathcal{M}'_d := \mathrm{PSL}_2(\mathbb{C}).\left(\mathrm{Rat}_d(\mathbb{R})\right)/\mathrm{PSL}_2(\mathbb{C})$$

$$\mathcal{M}'_d \subset \mathcal{M}_d(\mathbb{C})$$

Main Example: The Moduli Space $\mathcal{M}_2$

- The moduli space of quadratic rational maps can be identified with the plane $\mathbb{C}^2$ [Milnor 1993] via

$$\langle f \rangle \mapsto (\sigma_1 = \lambda_1 + \lambda_2 + \lambda_3, \sigma_2 = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1);$$

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$$\lambda_1, \lambda_2, \lambda_3 : \text{the multipliers of fixed points (Möbius invariants)}$$

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- $\mathcal{M}_2' \cong \mathbb{R}^2$ is the underlying real (σ_1, σ_2)-plane.

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Example

Maps $z \mapsto \frac{1}{\mu}\left(z \pm \frac{1}{z}\right)$ $(\mu \in \mathbb{R} - \{0\})$ are conjugate over $\mathbb{C}$

$$\frac{1}{\mathrm{i}} \cdot \frac{1}{\mu}\left(\mathrm{i}z + \frac{1}{\mathrm{i}z}\right) = \frac{1}{\mu}\left(z - \frac{1}{z}\right);$$

but restricted to $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$:

- $x \mapsto \frac{1}{\mu}\left(x - \frac{1}{x}\right)$ is a two-sheeted covering and of entropy $\log(2)$;
- $x \mapsto \frac{1}{\mu}\left(x + \frac{1}{x}\right)$ is of entropy zero.

Excluding Symmetries

This issue of real rational maps that are conjugate only over complex numbers can happen only in presence of *Möbius symmetries*; e.g.

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So **we have to exclude the *symmetry locus***

$$\mathcal{S}' := \{\langle f\rangle \in \mathcal{M}_d(\mathbb{C}) \mid f \in \mathrm{Rat}_d(\mathbb{R}), \mathrm{Aut}(f) \neq \{1\}\} \subset \mathcal{M}'_d$$

in order to have a well defined real entropy function.

The *Real Entropy Function* $h_{\mathbb{R}}$

Proposition (F. 2018)

For any $d \geq 2$:

- $\mathcal{M}'_d - \mathcal{S}'$ is an irreducible real variety of dimension $2d - 2$.
- The function

$$\begin{cases} h_{\mathbb{R}} : \mathcal{M}'_d - \mathcal{S}' \to [0, \log(d)] \\ \langle f \rangle \mapsto h_{\text{top}} \left(f \restriction_{\hat{\mathbb{R}}} : \hat{\mathbb{R}} \to \hat{\mathbb{R}} \right) \end{cases} \qquad (f \in \mathbb{R}(z))$$

  is surjective and continuous (in the analytic topology).
- The domain is disconnected with components (each of the same real dimension $2d - 2$) corresponding to possible topological degrees of the restriction $f \restriction_{\hat{\mathbb{R}}} : \hat{\mathbb{R}} \to \hat{\mathbb{R}}$.

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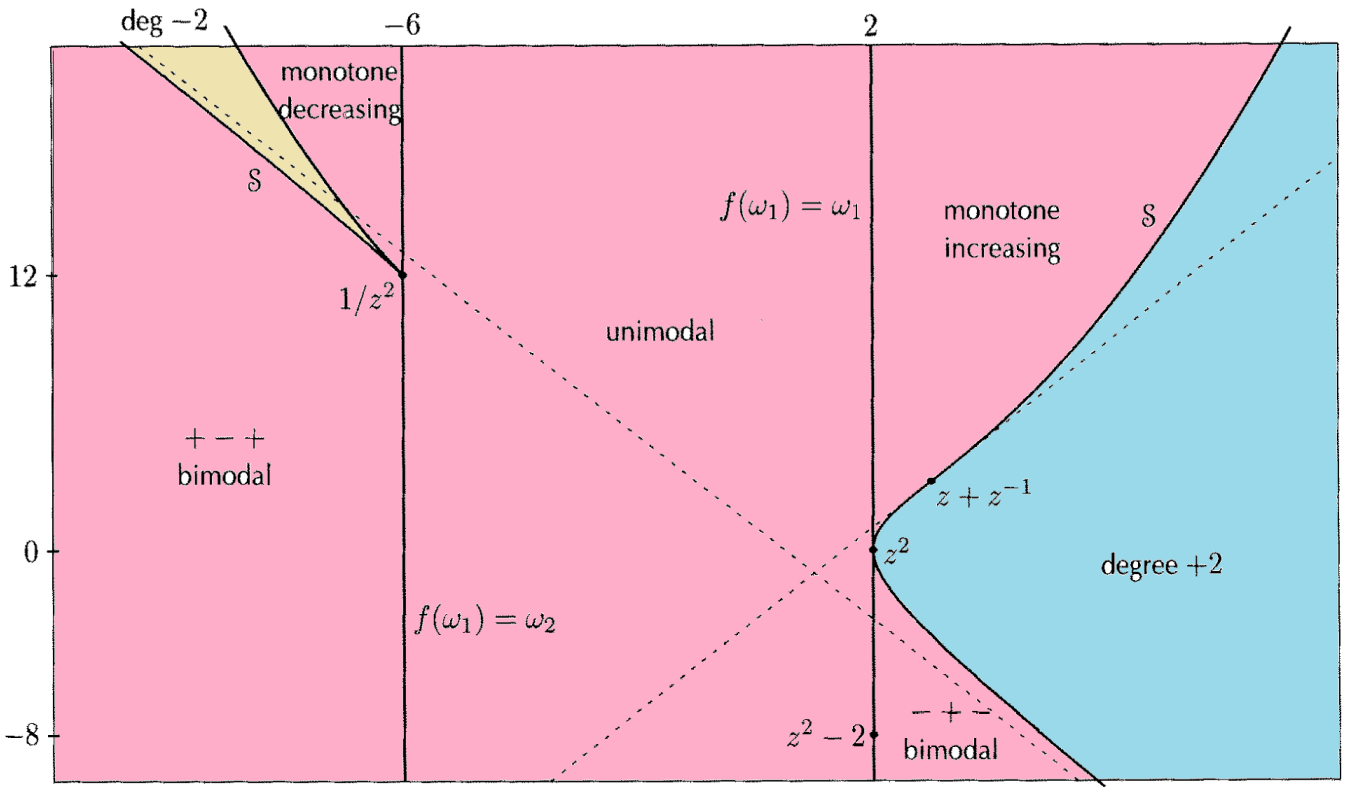
The *Monotonicity* Question

Restricted to connected components of the domain, are the level sets of $h_{\mathbb{R}}$ connected?

Back to the Main Example: $\mathcal{M}'_2 - \mathcal{S}'$ has three connected components.



$$\mathcal{S}' = \left\{ \left\langle \frac{1}{\mu}\left(z + \frac{1}{z}\right) \right\rangle = \left\langle \frac{1}{\mu}\left(z - \frac{1}{z}\right) \right\rangle \middle| \mu \in \mathbb{R} - \{0\} \right\} \text{ a cubic curve in } \mathcal{M}'_2 = \mathbb{R}^2$$

The topological degree of the restriction is ± 2 or zero.

A Simple Dichotomy for Real Quadratic Maps

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For $f \in \mathbb{R}(z)$ of degree two; either

- the two critical points are complex conjugate $\Rightarrow f\restriction_{\hat{\mathbb{R}}}: \hat{\mathbb{R}} \to \hat{\mathbb{R}}$ a 2-sheeted covering $\Rightarrow h_{\mathbb{R}}(f) = \log(2)$
- or the critical points are real; $f(\hat{\mathbb{R}}) \subsetneq \hat{\mathbb{R}} \Rightarrow$ only the interval map $f\restriction_{f(\hat{\mathbb{R}})}: f(\hat{\mathbb{R}}) \to f(\hat{\mathbb{R}})$ matters dynamically.

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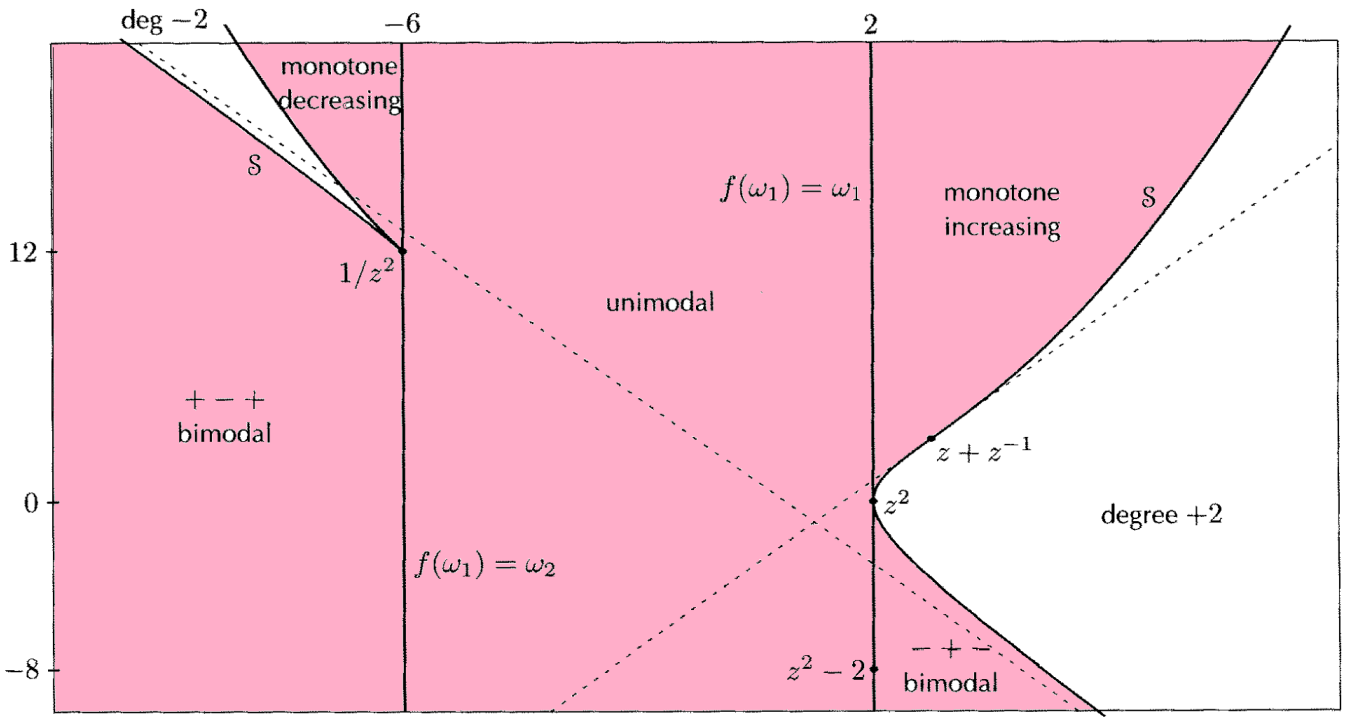
Upshot $\star$ Among all three connected components of $\mathcal{M}_2' - \mathcal{S}'$ only one connected component is relevant to our discussion; the component of degree zero maps.

The Component of Degree Zero Maps in $\mathcal{M}_2' - \mathcal{S}'$



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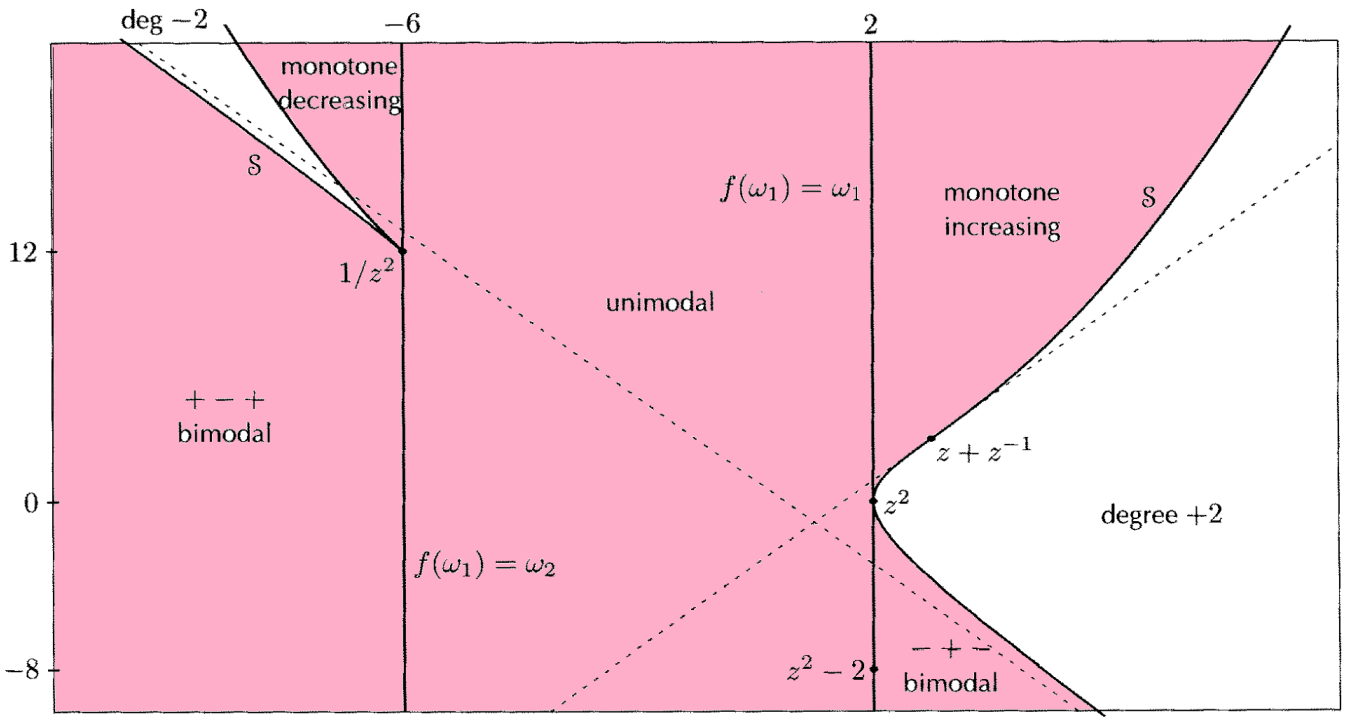

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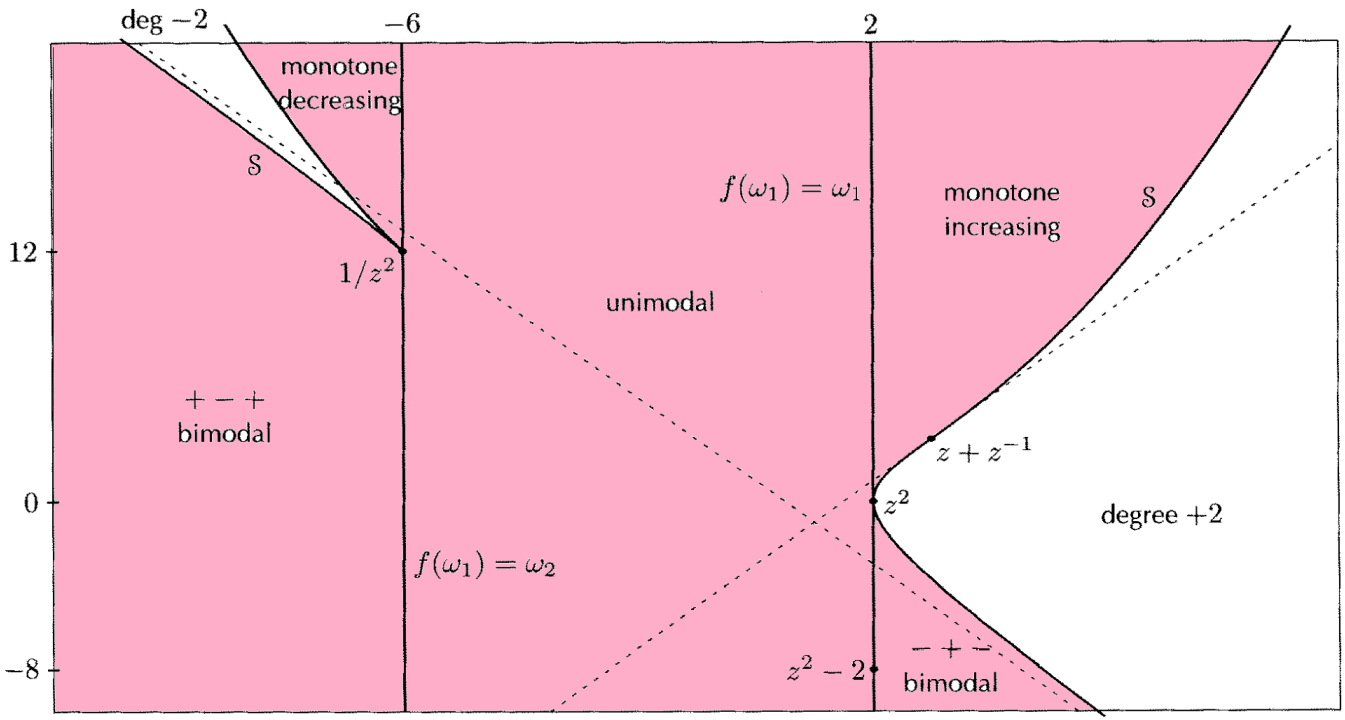
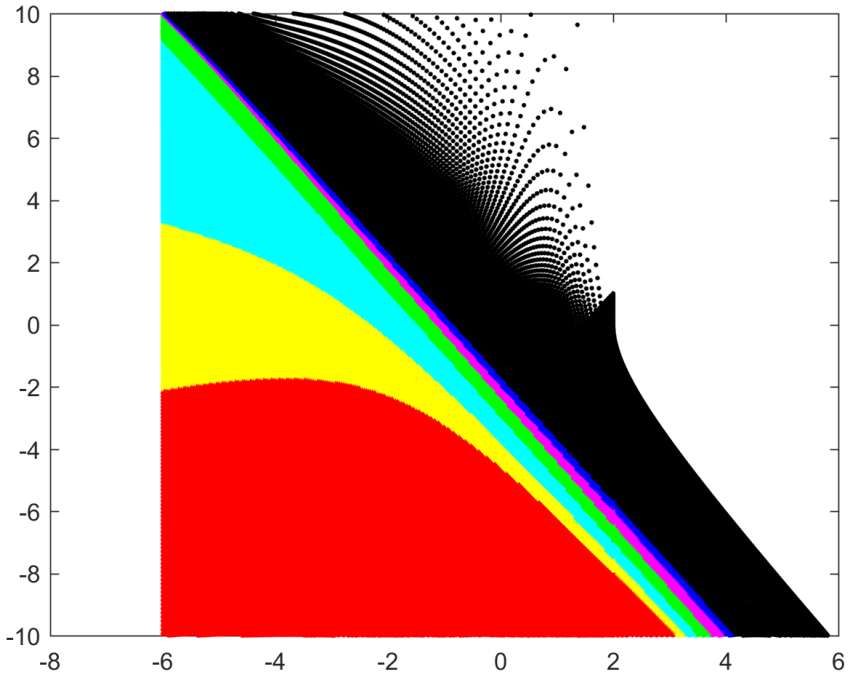


- There is a finer partition of this component according to the orientation and modality of the interval map.
- $h_{\mathbb{R}} \equiv 0$ on monotonic regions and $h_{\mathbb{R}} \equiv \log(2)$ on deg ± 2 regions.
- **Upshot $\star$ Only the unimodal region and the two bimodal regions adjacent to it matter to the monotonicity discussion.**

An Entropy Contour Plot in the Unimodal and $(-+-)$-Bimodal Regions



black	blue	magenta	green	cyan	yellow	red
$[0, 0.1)$	$[0.1, 0.25)$	$[0.25, 0.4)$	$[0.4, 0.48)$	$[0.48, 0.55)$	$[0.55, 0.65)$	$[0.65, \log(2) \approx 0.7]$

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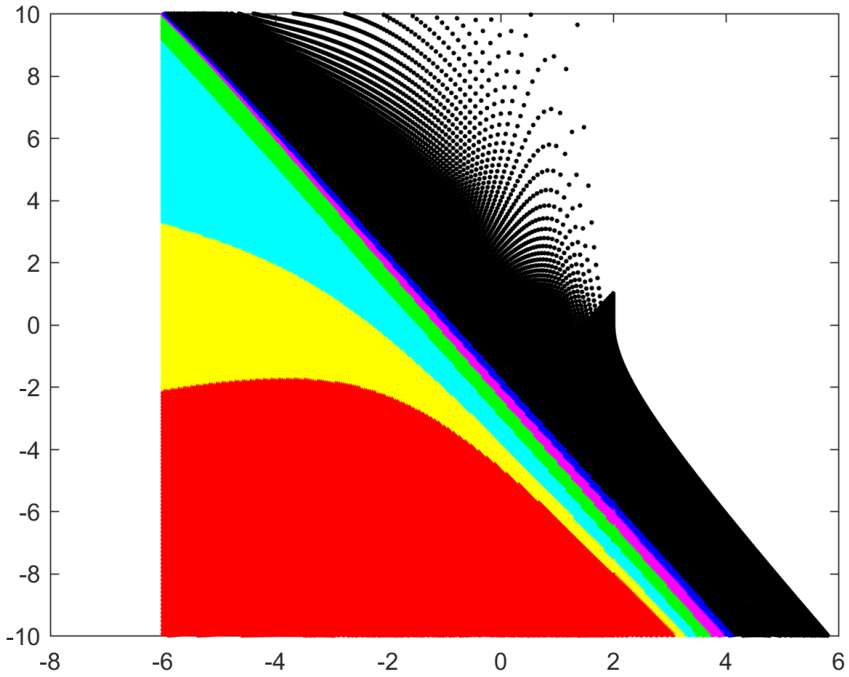

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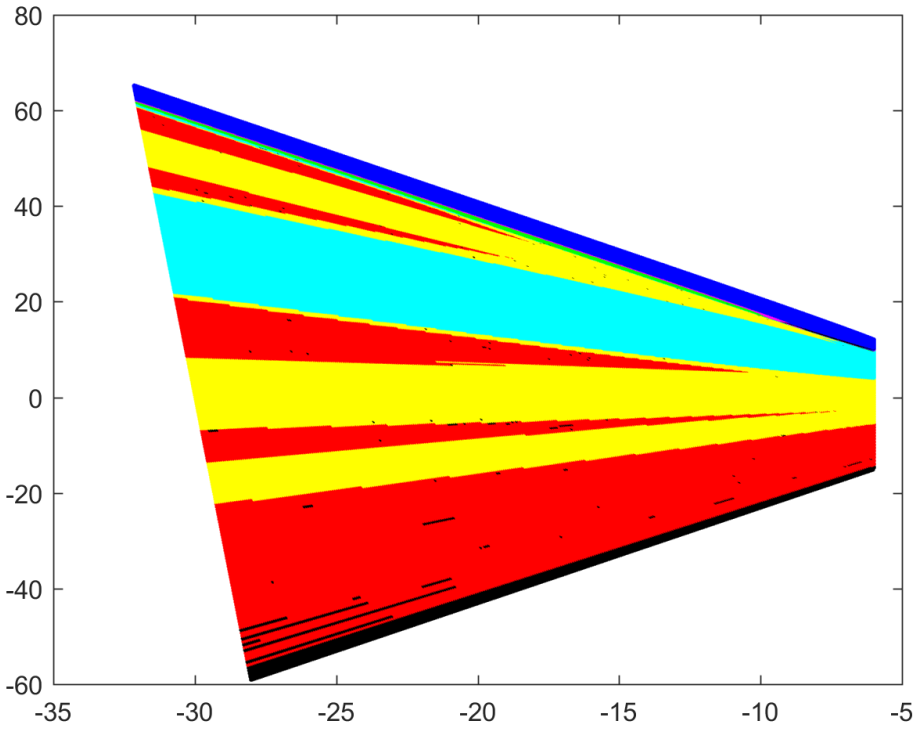
Conjecture

Restricted to the union of adjacent unimodal and $(-+-)$-bimodal regions the entropy function is monotonic.

An Entropy Contour Plot in the $(+-+)$-Bimodal Region



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failure	$[0, 0.05)$	$[0.05, 0.2)$	$[0.2, 0.3)$	$[0.3, 0.5)$	$[0.5, 0.66)$	$[0.66, \log(2) \approx 0.7]$

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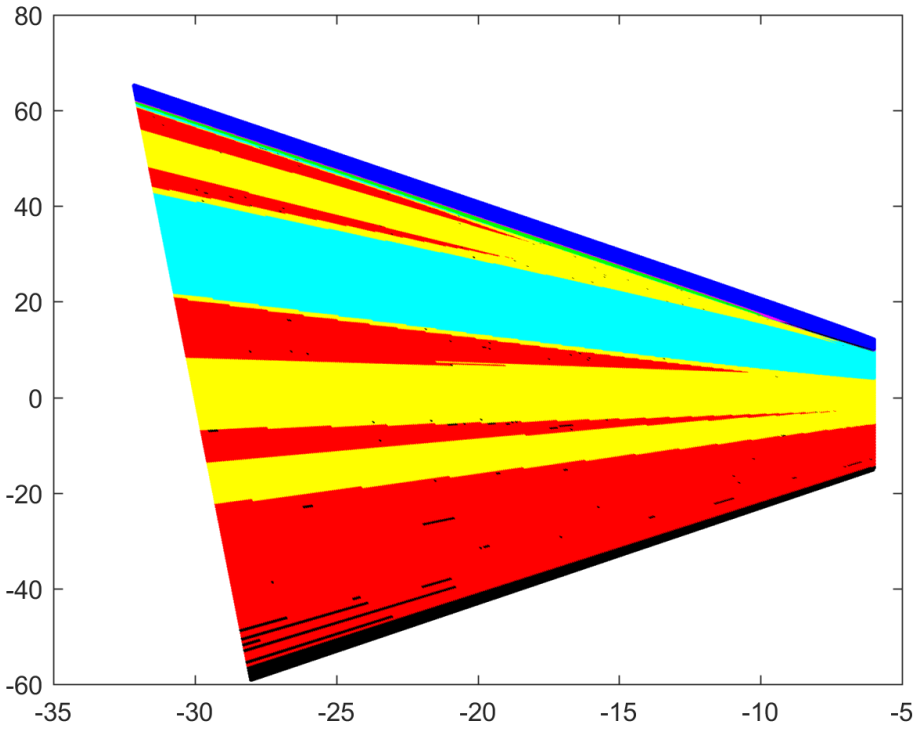

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Conjecture

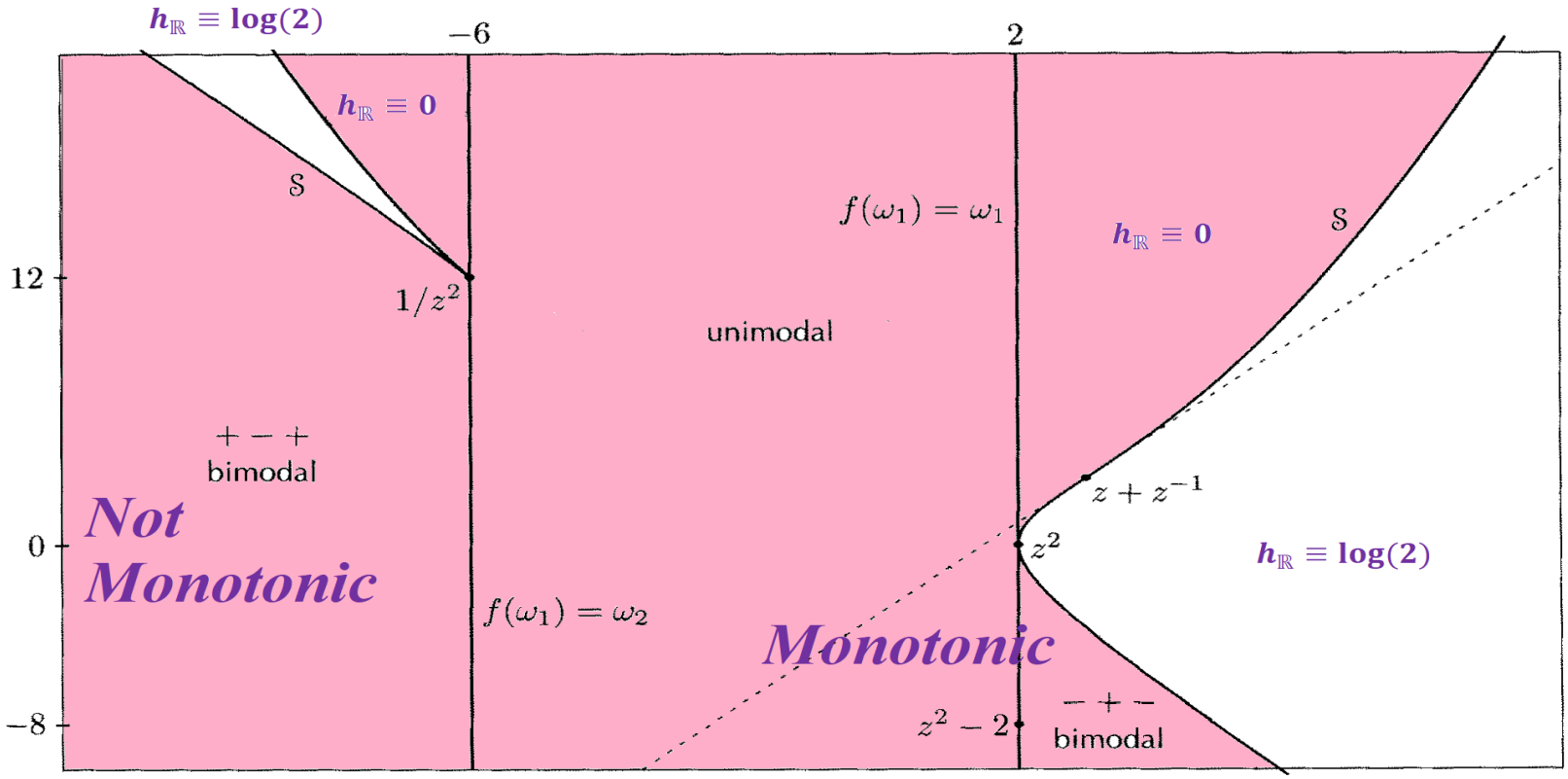
The monotonicity fails here due to a "non-polynomial" behavior.

The Algorithms Used to Generate the Plots

- L. Block, J. Keesling, S. Li, and K. Peterson. **An improved algorithm for computing topological entropy.** *J. Statist. Phys.*, 1989.
- L. Block and J. Keesling. **Computing the topological entropy of maps of the interval with three monotone pieces.** *J. Statist. Phys.*, 1992.

Going back to the moduli space

- Summarizing the conjectures:



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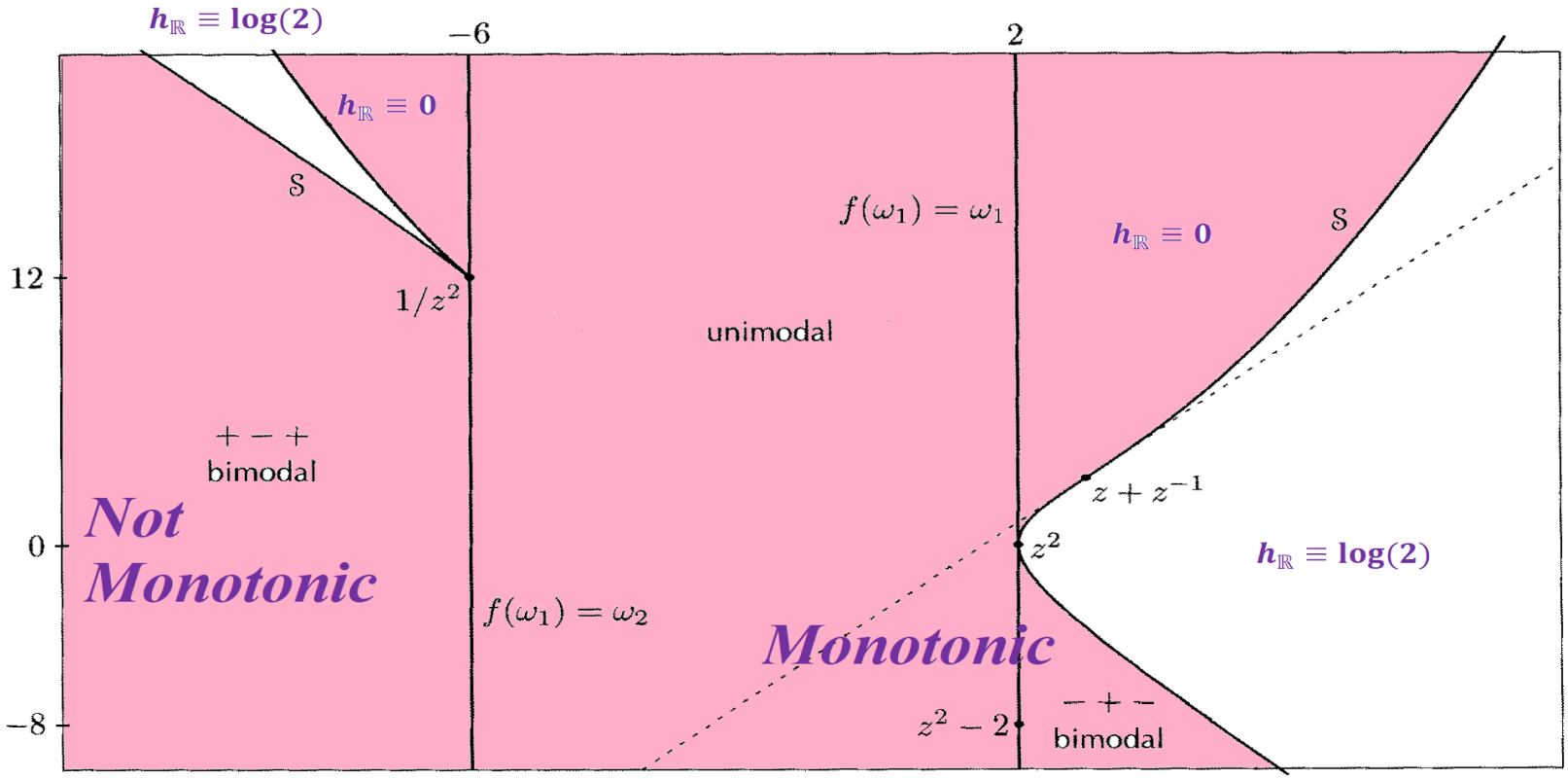


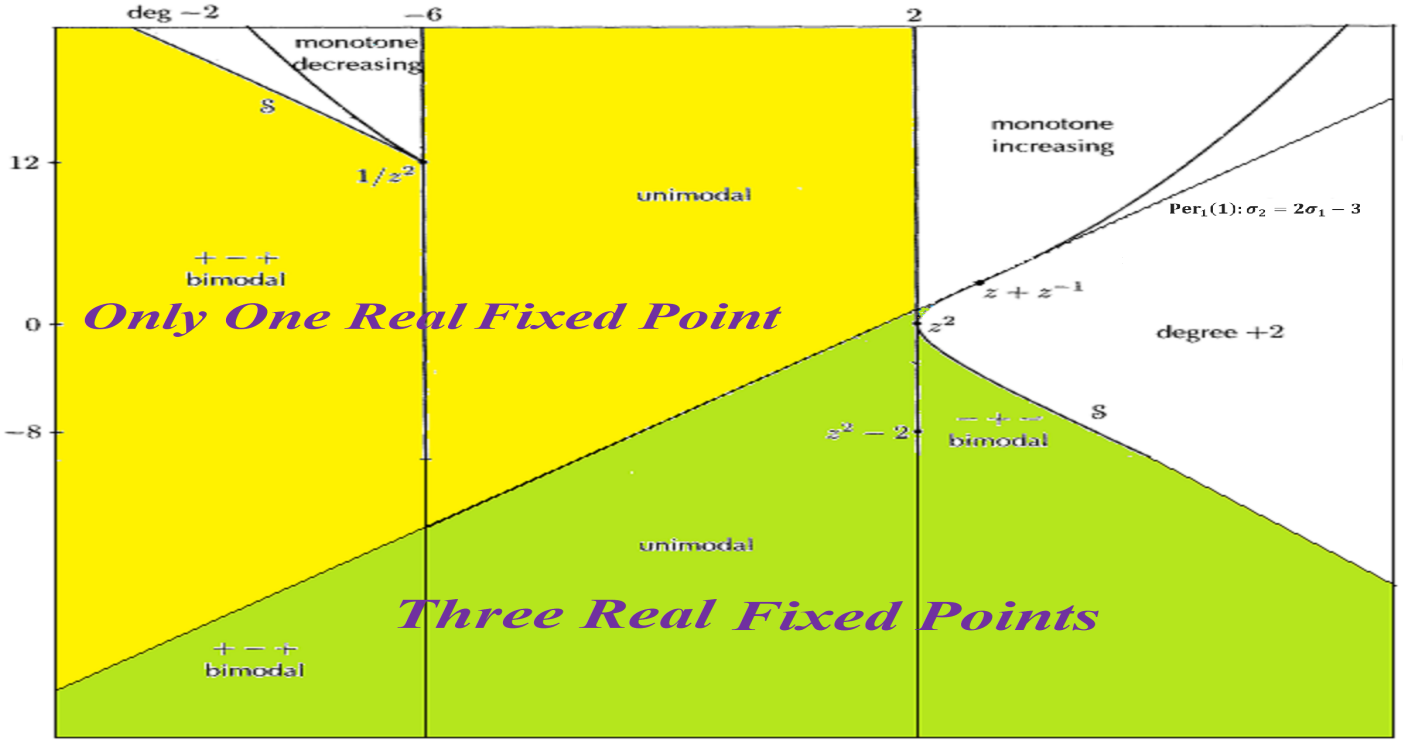
- An important line in the picture: $\sigma_2 = 2\sigma_1 - 3$ – the locus where one of the fixed points becomes multiple.

The Statement of the Theorem

Theorem (F. 2018)

Restricted to the part of the moduli space where the critical points are real and the maps admit three real fixed points, the level sets of $h_{\mathbb{R}}$ are connected.

Proof; Step 1: An Analysis of Real Fixed Points



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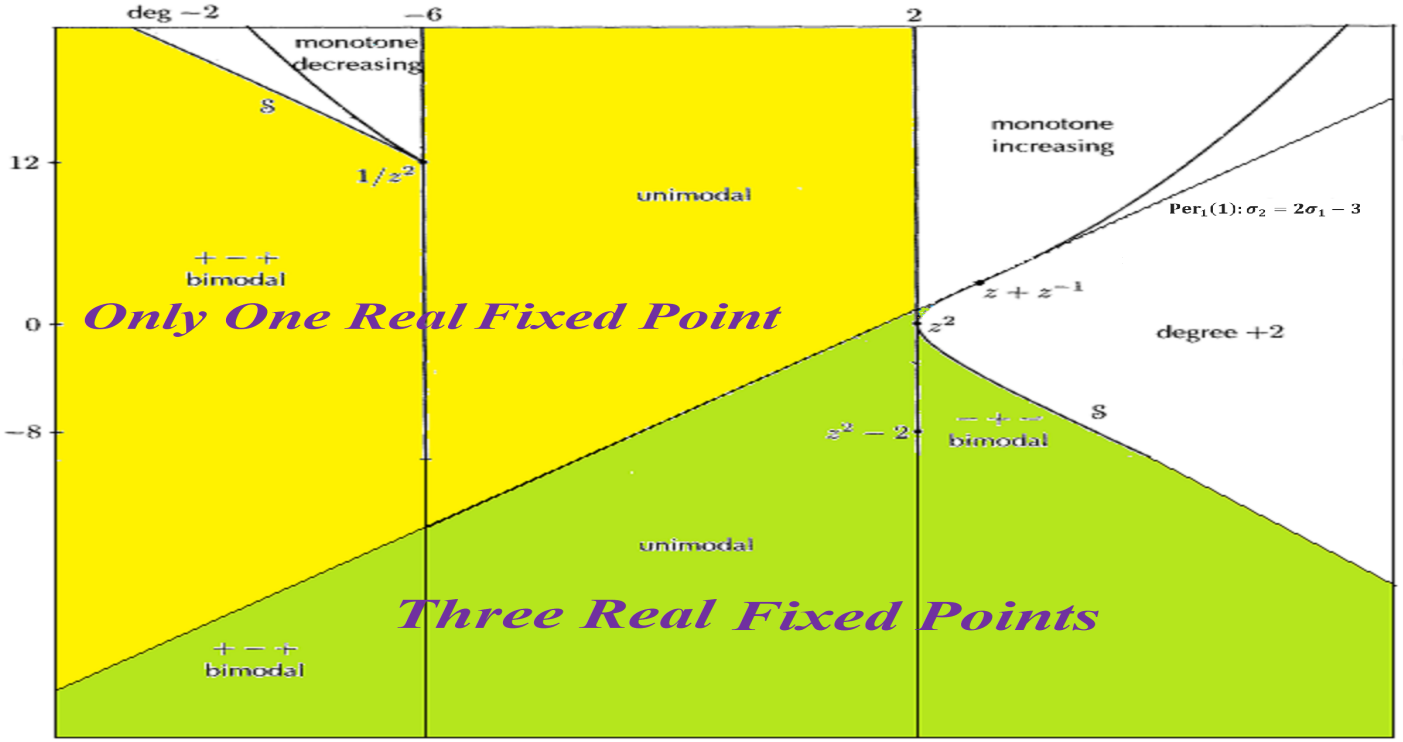

- If there are three real fixed points, at least one of them must be attracting:

$$\left.\begin{array}{l}\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R} - \{1\} \text{ w/ at least one of them non-negative} \\ \frac{1}{1-\lambda_1} + \frac{1}{1-\lambda_2} + \frac{1}{1-\lambda_3} = 1\end{array}\right\} \Rightarrow \exists i \text{ s.t. } |\lambda_i| < 1.$$

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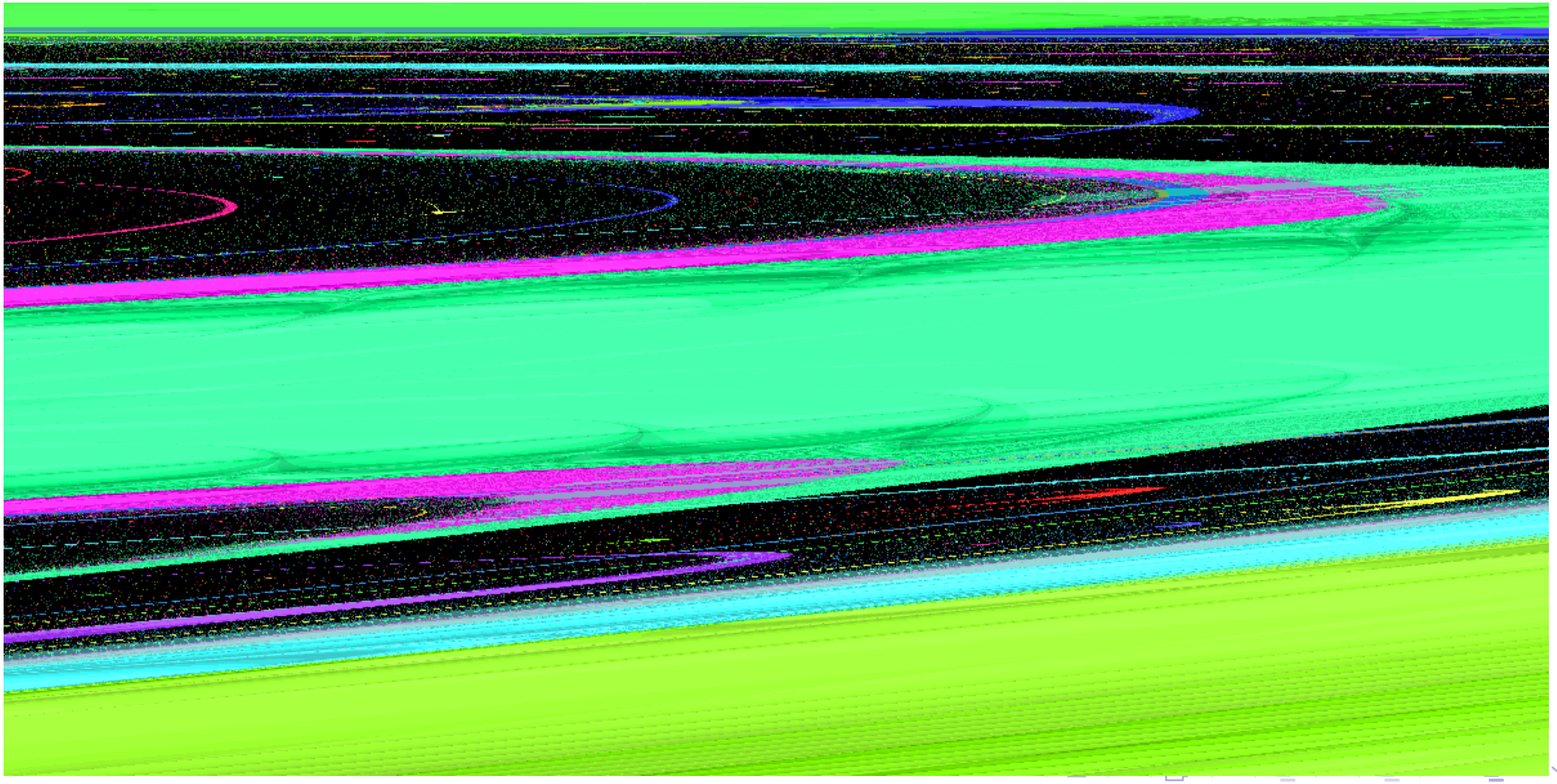
- This straightening can be done through the family.
- The monotonicity of entropy for quadratic polynomials has been established by Milnor and Thurston.

An Interesting Bifurcation Behavior

The bifurcation diagram for a part of the $(+-+)$-bimodal region parametrized as $\left\{x \mapsto \frac{2\mu x(tx+2)}{\mu^2 x^2+(tx+2)^2} : [-1,1] \circlearrowleft\right\}_{-26<\mu<-19,-5<t<-1}$. A period-doubling bifurcation from a 5-cycle to a 10-cycle visible as the transition from green to magenta occurs in “various” directions.



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- Why "non-polynomial"?
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- The main idea of the proof:
  Construct a family of real hyperbolic components consisting of essentially non-polynomial quadratic rational maps in the $(+-+)$-region.